Trigonometric Calogero-Moser System as a Symmetry Reduction of KP Hierarchy

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Abstract

Trigonometric non-isospectral flows are defined for KP hierarchy. It is demonstrated that symmetry constraints of KP hierarchy associated with these flows give rise to trigonometric Calogero-Moser system.

1 Introduction

This paper may be considered as a sequel of the work [1], where it was shown that rational Calogero-Moser system can be obtained by a symmetry constraint of KP hierarchy. Here we show that a simple generalization of the scheme leads to *trigonometric* Calogero-Moser system. We describe the corresponding symmetries and symmetry constrains in the framework of analytic-bilinear approach to integrable hierarchies [2, 3, 4] (the primary objects in this approach are Cauchy-Baker-Akhiezer (CBA) function and Hirota bilinear identity for it), as well as in terms of free fermionic fields [5].

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2 Hirota Identity and KP Hierarchy

First we give a sketch of the picture of generalized KP hierarchy in frame of analytic-bilinear approach; for details we refer to [2, 3, 4].

The formal starting point is Hirota bilinear identity for Cauchy-Baker-Akhiezer function,

$$\oint \chi(\nu,\mu;g_1)g_1(\nu)g_2^{-1}(\nu)\chi(\lambda,\nu;g_2)d\nu = 0 \quad \lambda\,,\mu\in D. \tag{1}$$

Here $\chi(\lambda,\mu;g)$ (the Cauchy kernel) is a function of two complex variables $\lambda,\mu\in\bar{D}$, where D is a unit disc, and a functional of the loop group element $g\in\Gamma^+$, i.e., of a complex-valued function analytic and having no zeros in $\mathbf{C}\setminus D$, equal to 1 at infinity; the integration goes over the unit circle . By definition, the function $\chi(\lambda,\mu)$ possesses the following analytical properties: as $\lambda\to\mu$, $\chi\to(\lambda-\mu)^{-1}$ and $\chi(\lambda,\mu)$ is an analytic function of two variables $\lambda,\mu\in\bar{D}$ for $\lambda\neq\mu$. The function $\chi(\lambda,\mu;g)$ is a solution to (1) if it possesses specified analytic properties and satisfies (1) for all $\lambda,\mu\in D$ and some class of loops $g\in\Gamma^+$.

Parametrization of Γ^+ in terms of standard KP variables

$$g(\lambda)=g(\lambda,\mathbf{x})=\exp\left(\sum_{n=1}^{\infty}\lambda^{-n}x_n\right)$$

gives an opportunity to consider functionals of $g\in\Gamma^+$ as functions of an infinite set of KP variables $\mathbf{x}$.

In another form, more similar to standard Hirota bilinear identity, the identity (1) can be written as

$$\oint \psi(\lambda,\nu;g_2)\psi(\nu,\mu;g_1)d\lambda = 0, \tag{2}$$

where

$$\psi(\lambda,\mu,g)=g(\lambda)\chi(\lambda,\mu,g)g^{-1}(\mu).$$

We call the function $\psi(\lambda,\mu;g)$ a Cauchy-Baker-Akhiezer function.

Hirota bilinear identity (1) incorporates the standard Hirota bilinear identity for the Baker-Akhiezer (BA) and dual (adjoint) Baker-Akhiezer function of the KP hierarchy. Indeed, let us introduce these functions by the formulae

$$\psi(\lambda;g)=g(\lambda)\chi(\lambda;0),$$
$$\widetilde{\psi}(\mu;g)=g^{-1}(\mu)\chi(0;\mu).$$

Then for Baker-Akhiezer function $\psi(\lambda; g)$ and dual Baker-Akhiezer function $\widetilde{\psi}(\mu; g)$, taking the identity (1) at $\lambda = \mu = 0$, we get the usual form of Hirota bilinear identity

$$\oint \widetilde{\psi}(\nu; g_2)\psi(\nu; g_1)d\nu = 0. \tag{3}$$

The only difference with the standard setting here is that we define BA and dual BA function in the neighborhood of zero, not in the neighborhood of infinity.

There are three different types of integrable equations implied by identity (1), that correspond to the KP hierarchy in the usual form (in terms of potentials), to the modified KP hierarchy and to the hierarchy of the singular manifold equations. They arise for different types of functions connected with the Cauchy-Baker-Akhiezer function satisfying Hirota bilinear identity (see the derivation in [2], [3], [4]).

1.On the first level, we have hierarchy of equations for the diagonal of the regularized Cauchy kernel taken at zero (the potential)

$$\begin{aligned} &u(g) = \chi_r(0, 0; g), \\ &\chi_r(\lambda, \mu; g) = \chi(\lambda, \mu; g) - (\lambda - \mu)^{-1}. \end{aligned}$$

The first equation of this hierarchy is a potential form of KP equation

$$\partial_x \left(u_t - \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}(u_x)^2 \right) = \tfrac{3}{4}u_{yy}\,, \tag{4}$$

where $x = x_1$, $y = x_2$, $t = x_3$, which reduces to standard KP equation for the function $v = -2\partial_x u$.

2.On the second level, there are the equations for the Baker-Akhiezer and dual Baker-Akhiezer type wave functions (the modified equations)

$$\Psi(g) = \int \psi(\lambda, g)\rho(\lambda)d\lambda\,,$$

$$\widetilde{\Psi}(g) = \int \widetilde{\rho}(\mu)\widetilde{\psi}(\mu, g)d\mu\,.$$

3.On the third level we have the equations for the Cauchy-Baker-Akhiezer type wave function

$$\Phi(g) = \oint\oint (\psi_r(\lambda, \mu; g))\rho(\lambda)\widetilde{\rho}(\mu)d\lambda\, d\mu,$$

where $\rho(\lambda)$, $\widetilde{\rho}(\mu)$ are some arbitrary weight functions.

3 Nonisospectral Symmetries

The (isospectral) dynamics defined by Hirota bilinear identity (1) is connected with operator of multiplication by loop group element $g \in \Gamma^+$; this dynamics can be interpreted in terms of commuting flows corresponding to infinite number of 'times' x_n. A general idea of introduction additional (in general, non-commutative) symmetries is to consider more general operators $\hat{R}$ on the unit circle. Let us introduce symmetric bilinear form

$$(f|g) = \oint f(\nu)g(\nu)d(\nu).$$

In terms of this form identity (1) looks like

$$(\chi(\dots,\mu;g_1)g_1(\dots)|g_2^{-1}(\dots)\chi(\lambda,\dots;g_2)) = 0 \quad \lambda\,,\mu \in D, \tag{5}$$

or, for Cauchy-Baker-Akhiezer function $\psi(\lambda,\mu;g)$,

$$(\psi(\dots,\mu;g_1)|\psi(\lambda,\dots;g_2)) = 0 \quad \lambda\,,\mu \in D, \tag{6}$$

where by dots we denote the argument which is involved into integration. Let some CBA function $\psi(\lambda,\mu;g)$ satisfying Hirota bilinear identity be given. We define symmetry transformation connected with arbitrary invertible linear operator $\hat{R}$ in the space of functions on the unit circle by the equations

$$(\widetilde{\psi}(\dots,\mu;g_1)|\hat{R}|\psi(\lambda,\dots;g_2)) = 0,$$
$$(\psi(\dots,\mu;g_1)|\hat{R}^{-1}|\widetilde{\psi}(\lambda,\dots;g_2)) = 0.$$

It is possible to show that if both these equations for the transformed CBA function $\widetilde{\psi}(\lambda,\mu;g)$ are solvable, then the solution for them is the same (and unique), and it satisfies identity (6). In this case the symmetry transformation connected with operator $\hat{R}$ is correctly defined. It is also possible to define one-parametric groups of transformations by the equation

$$(\psi(\dots,\mu;g_1,\Theta_1)|\exp((\Theta_1-\Theta_2)\hat{r})|\psi(\lambda,\dots;g_2,\Theta_2)) = 0, \tag{7}$$

where Θ is a parameter (non-isospectral 'time'). Taking the generators $\hat{r}_{mn} = \lambda^{-m}\partial_\lambda^n$, we get noncommutative symmetries in the form proposed by Orlov and Shulman [7].

In the work [1] non-isospectral symmetries connected with operators with degenerate kernel were considered (similar symmetries were studied in [9]). In particular, generators with the kernel of the form

$$r_{\alpha\beta}(\nu,\nu') = 2\pi \mathrm{i}\delta(\alpha-\nu)\delta(\beta-\nu'), \tag{8}$$

where α, β belong to the unit circle, were used.

More general case of generators

$$r_{\rho\widetilde{\rho}}(\nu,\nu') = 2\pi \mathrm{i}\widetilde{\rho}(\nu')\rho(\nu), \tag{9}$$

was also studied, but for simplicity it was put

$$(\widetilde{\rho}|\rho) = 0.$$

The crucial point for this work is to *generalize this condition*, and consider degenerate operators with nonzero pairing of factors

$$2\pi \mathrm{i}(\widetilde{\rho}|\rho) = h,$$

or, more generally,

$$\begin{aligned} r_h(\nu,\nu') &= 2\pi \mathrm{i}\sum_{i=1}^{N} \widetilde{\rho}_i(\nu')\rho_i(\nu), \\ 2\pi \mathrm{i}(\widetilde{\rho}_i|\rho_j) &= h\delta_{ij}, \end{aligned} \tag{10}$$

where h is some constant. We will show that generators of these form define trigonometric nonisospectral flows.

3.1 Trigonometric Flows

Using simple identity

$$\hat{r}_h^2 = h\hat{r}_h,$$

we get the formula

$$\exp(\Theta\hat{r}_h) = 1 + \hat{r}_h\frac{e^{\Theta h}-1}{h}. \tag{11}$$

Then, performing integration in the equation (7) taken for $g_1 = g_2$, which in this case reads

$$\oint\oint d\nu d\nu' \psi(\nu,\mu;g,\Theta_1)(\delta(\nu-\nu') + \frac{e^{(\Theta_1-\Theta_2)h}-1}{h} 2\pi \mathrm{i} \sum_{i=1}^{N} \widetilde{\rho}_i(\nu')\rho_i(\nu))$$
$$\times\psi(\lambda,\nu';g,\Theta_2)) = 0, \quad (12)$$

we get equation for the CBA function

$$\psi(\lambda,\mu;\mathbf{x},\Theta+\Delta\Theta) = \psi(\lambda,\mu;\mathbf{x},\Theta)$$
$$+\frac{e^{h\Delta\Theta}-1}{h}\sum_{i=1}^{N}\widetilde{\phi}_i(\lambda;\mathbf{x},\Theta)\phi_i(\mu;\mathbf{x},\Theta+\Delta\Theta), \quad (13)$$

or, in differential form,

$$\partial_\theta\psi(\lambda,\mu;\mathbf{x},\Theta) = \sum_{i=1}^{N}\widetilde{\phi}_i(\lambda;\mathbf{x},\Theta)\phi_i(\mu;\mathbf{x},\Theta), \quad (14)$$

where

$$\widetilde{\phi}_i(\lambda;\mathbf{x},\Theta) = \oint d\mu\psi(\lambda,\mu;\mathbf{x},\Theta)\widetilde{\rho}(\mu),$$
$$\phi_i(\mu;\mathbf{x},\Theta) = \oint d\lambda\psi(\lambda,\mu;\mathbf{x},\Theta)\rho(\lambda).$$

It is possible to resolve equation (13) and express $\psi(\lambda,\mu;\mathbf{x},\Theta_{\alpha\beta}+\Delta\Theta_{\alpha\beta})$ through $\psi(\lambda,\mu;\mathbf{x},\Theta_{\alpha\beta})$. First we integrate equation (13) with the weight function $\rho_k(\lambda)$ and get

$$\phi_k(\mu;\mathbf{x},\Theta+\Delta\Theta) = \phi_k(\mu;\mathbf{x},\Theta)$$
$$+\frac{e^{h\Delta\Theta}-1}{h}\sum_{i=1}^{N}\Phi_{ki}(\mathbf{x},\Theta)\phi_i(\mu;\mathbf{x},\Theta+\Delta\Theta), \quad (15)$$

where

$$\Phi_{ij}(g) = \oint\oint(\psi(\lambda,\mu;g)-(\lambda-\mu)^{-1})\rho_i(\lambda)\widetilde{\rho}_j(\mu)d\lambda\, d\mu$$
$$-2\pi\mathrm{i}(\rho_i^{\mathrm{out}}|\widetilde{\rho}_j^{\mathrm{in}}),$$

and

$$
\begin{aligned}
f^{\mathrm{in}}(\lambda) &= \frac{1}{2\pi \mathrm{i}} \oint d\mu f(\mu)(\mu - \lambda^{\mathrm{in}})^{-1} \\
&= \frac{1}{2\pi \mathrm{i}} \mathrm{v.p.} \oint d\mu f(\mu)(\mu - \lambda)^{-1} + \frac{1}{2} f(\lambda), \\
f^{\mathrm{out}}(\lambda) &= -\frac{1}{2\pi \mathrm{i}} \oint d\mu f(\mu)(\mu - \lambda^{\mathrm{out}})^{-1} \\
&= -\frac{1}{2\pi \mathrm{i}} \mathrm{v.p.} \oint d\mu f(\mu)(\mu - \lambda)^{-1} + \frac{1}{2} f(\lambda).
\end{aligned}
$$

In matrix form,

$$
\begin{aligned}
&|\phi(\mu; \mathbf{x}, \Theta + \Delta\Theta)\rangle \\
&\qquad = |\phi(\mu; \mathbf{x}, \Theta)\rangle + \frac{e^{h\Delta\Theta} - 1}{h} \Phi(\mathbf{x}, \Theta) |\phi(\mu; \mathbf{x}, \Theta + \Delta\Theta)\rangle.
\end{aligned}
$$

Resolving this equation with respect to $|\phi(\Theta + \Delta\Theta)\rangle$, we obtain

$$
|\phi(\mu; \mathbf{x}, \Theta + \Delta\Theta)\rangle = (I - \frac{e^{h\Delta\Theta} - 1}{h} \Phi(\mathbf{x}, \Theta))^{-1} |\phi(\mu; \mathbf{x}, \Theta)\rangle. \tag{16}
$$

Substituting (16) into (13), we finally get

$$
\begin{aligned}
&\psi(\lambda, \mu; \mathbf{x}, \Theta + \Delta\Theta) = \psi(\lambda, \mu; \mathbf{x}, \Theta) \\
&\quad + \frac{e^{h\Delta\Theta} - 1}{h} \langle \widetilde{\phi}(\lambda; \mathbf{x}, \Theta) | (I - \frac{e^{h\Delta\Theta} - 1}{h} \Phi)^{-1} |\phi(\mu; \mathbf{x}, \Theta)\rangle.
\end{aligned} \tag{17}
$$

The formula (17) explicitly defines *discrete nonisospectral symmetry* of KP hierarchy in terms of CBA function.

In particular, this formula expresses the function $\psi(\lambda, \mu; \mathbf{x}, \Theta)$ through the initial data $\psi_0(\lambda, \mu; \mathbf{x}) = \psi(\lambda, \mu; \mathbf{x}, \Theta = 0)$, thus giving explicit formula for the action of non-isospectral flow connected with the generator (10) on the CBA function. This flow appears to be *trigonometric*, because the CBA function and other objects of the hierarchy defined through it (potential, wave functions) depend rationally on $\exp(h\Theta)$.

Using the formula (17), it is also possible to get the action of the trigonometric flow on the τ-function. Using simple identity

$$
\det(I + |f\rangle\langle g|) = \langle g|f\rangle
$$

and determinant formula for the transformation of CBA function under the action of a rational loop (see [4]),

$$\psi_0(\alpha, \beta; \mathbf{x} + [\mu] - [\lambda]) = \frac{\det \begin{pmatrix} \psi_0(\lambda, \mu; \mathbf{x}) & \psi_0(\lambda, \beta; \mathbf{x}) \\ \psi_0(\alpha, \mu; \mathbf{x}) & \psi_0(\alpha, \beta; \mathbf{x}) \end{pmatrix}}{\psi_0(\lambda, \mu; \mathbf{x})}, \tag{18}$$

we get another representation of the formula (17),

$$\psi(\lambda, \mu; \mathbf{x}, \Theta) = \psi_0(\lambda, \mu; \mathbf{x}) \frac{\det(I - \frac{e^{h\Theta}-1}{h}\Phi_0(\mathbf{x} + [\mu] - [\lambda]))}{\det(I - \frac{e^{h\Theta}-1}{h}\Phi_0(\mathbf{x}))}. \tag{19}$$

Comparing this formula with the formula connecting the CBA function and the τ-function (which in fact defines the τ-function through the CBA function)

$$\psi(\lambda, \mu, \mathbf{x}) = g(\lambda, \mathbf{x}) g(\mu, \mathbf{x})^{-1} \frac{1}{\lambda - \mu} \frac{\tau(\mathbf{x} + [\mu] - [\lambda])}{\tau(\mathbf{x})}, \tag{20}$$

we come to the conclusion that the τ-function corresponding to the transformed CBA function $\psi(\lambda, \mu; \mathbf{x}, \Theta)$ is given by the expression

$$\tau(\mathbf{x}, \Theta) = \tau_0(\mathbf{x}) \det \left(I - \frac{e^{h\Theta} - 1}{h} \Phi_0(\mathbf{x}) \right). \tag{21}$$

Thus we have explicitly defined action of non-isospectral symmetry with the generator (10) on KP τ-function. This formula also defines the evolution of KP potential $u(\mathbf{x})$,

$$\begin{aligned} u(\mathbf{x}, \Theta) = \psi_r(0, 0; \mathbf{x}, \Theta) &= -\partial_x \ln \tau(\mathbf{x}, \Theta) \\ &= u_0(\mathbf{x}) - \partial_x \ln \det \left(I - \frac{e^{h\Theta} - 1}{h} \Phi_0(\mathbf{x}) \right), \end{aligned} \tag{22}$$

where $x = x_1$ (it is easy to get this formula directly from (17)).

3.2 Möbius-type Symmetry

The transformation of the matrix Φ under the action of trigonometric non-isospectral flow is especially simple. According to (17), it looks like

$$\Phi(\mathbf{x}, \Theta) = \frac{\Phi_0(\mathbf{x}) e^{h\Theta}}{1 - \frac{e^{h\Theta}-1}{h}\Phi_0(\mathbf{x})}, \tag{23}$$

and it is nothing more than matrix Möbius-type transformation. Differential equation defining this transformation is

$$\frac{\partial \Phi(\mathbf{x}, \Theta)}{\partial \Theta} = \Phi^2(\mathbf{x}, \Theta) + h\Phi(\mathbf{x}, \Theta) \tag{24}$$

The difference with the work [1] is that we consider generic Möbius-type one-parametric flow, which is trigonometric.

We would like to recall (see [1]) that transformation of the KP potential u corresponding to matrix inversion Φ^{-1} looks like

$$u(\Phi^{-1}) = u(\Phi) - \partial_x \ln \det \Phi, \tag{25}$$

and it represents a composition formula for several binary Bäcklund transformations (this formula can also be derived from (22), (23) in the limit $\Theta \to \infty$). Taking into account that the transformations $\Phi + C$, $A\Phi B$, where A, B, C are constant matrices, correspond to identical transformation of the potential, the formula (25) is sufficient to define the transformation of potential corresponding to arbitrary matrix Möbius transformation of Φ. Thus, to the derive formula (22), it is enough to fix the generator of one-parametric subgroup of the Möbius group (24) and to use the formula (25).

4 Symmetry Constraints and Calogero-Moser System

Now, when we have identified trigonometric nonisospectral flows, it is quite straightforward to interpret Calogero-Moser system as a symmetry constraint of KP hierarchy.

Proposition 4.1 *Let us impose the following symmetry constraint:*

$$\partial_\Theta u(\mathbf{x}, \Theta) = \partial_x u(\mathbf{x}, \Theta). \tag{26}$$

Then the dependence $u(\mathbf{x}, \Theta)$ on x is trigonometric, and the motion of poles of $u(\mathbf{x}, \Theta)$ in the x-plane with respect to the 'time' $y = x_2$ is described by trigonometric Calogero-Moser system.

Proof. The dependence of KP potential on Θ is explicitly given by the formula (22) and it is trigonometric (rational in $\exp(h\Theta)$). Thus a constraint enforces trigonometric dependence of $u(\mathbf{x},\Theta)$ on x, that, according to [12], leads to trigonometric Calogero-Moser system

$$\partial_y^2 x^i = 4\sum_{j\neq i} V'(x^i - x^j), \quad V(x) = \frac{1}{\sinh^2 x}, \quad V'(x) = \partial_x V(x). \tag{27}$$

Solutions to this system, due to the formula (22), are defined through the eigenvalues of the matrix $\Phi(\mathbf{x})$.

We would like to make a remark clarifying the relation between constraint (26) and standard constrained KP hierarchy (see [10], [11]), which is defined by the condition

$$\partial_x u(\mathbf{x}) = \sum_{i=1}^{N} \Psi_i(\mathbf{x})\widetilde{\Psi}_i(\mathbf{x}), \tag{28}$$

and represents multicomponent AKNS-type system for the wave functions. The KP potential $u(\mathbf{x},\Theta)$ defined by the formula (22) satisfies differential equation

$$\partial_\Theta u(\mathbf{x},\Theta) = \partial_x \mathrm{tr}|\Phi(\mathbf{x},\Theta)| = \sum_{i=1}^{N} \Psi_i(\mathbf{x},\Theta)\widetilde{\Psi}_i(\mathbf{x},\Theta) \tag{29}$$

(it is easy to check it directly or derive from (14)). Thus, the standard constraint is given by the condition (26) taken at the single point $\Theta = 0$ (or at some fixed point). Therefore the constraint (26) is a stronger constraint, and potentials $u(\mathbf{x},\Theta)$ satisfying this constraint also satisfy condition (28) (for all Θ).

5 Fermionic Fields and Calogero-Moser System

The results we have presented can be reformulated using the KP theory in terms of fermionic fields [5]. Now let us review some facts from this theory. We have fermionic fields

$$\psi(z) = \sum_{k\in Z} \psi_k z^k, \quad \psi^*(z) = \sum_{k\in Z} \psi_k^* z^{-k-1},$$

where fermionic operators satisfy the canonical anti-commutation relations:

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0; \qquad [\psi_m, \psi_n^*]_+ = \delta_{mn}. \tag{30}$$

Let us introduce left and right vacuums by the properties:

$$\begin{array}{llll} \psi_m|0\rangle = 0 & (m < 0), & \psi_m^*|0\rangle = 0 & (m \geq 0), \\ \langle 0|\psi_m = 0 & (m \geq 0), & \langle 0|\psi_m^* = 0 & (m < 0). \end{array}$$

Note that the subscript $*$ does not denote the complex conjugation. The vacuum expectation value is defined by relations:

$$\langle 0|1|0\rangle = 1, \quad \langle 0|\psi_m \psi_m^*|0\rangle = 1 \quad m < 0, \quad \langle 0|\psi_m^* \psi_m|0\rangle = 1 \quad m \geq 0,$$

$$\langle 0|\psi_m \psi_n|0\rangle = \langle 0|\psi_m^* \psi_n^*|0\rangle = 0, \quad \langle 0|\psi_m \psi_n^*|0\rangle = 0 \quad m \neq n.$$

Let us denote $\widehat{gl}(\infty) = Lin\{1, : \psi_i \psi_j^* : |i, j \in Z\}$, with usual normal ordering $: \psi_i \psi_j^* := \psi_i \psi_j^* - \langle 0|\psi_i \psi_j^*|0\rangle$. We define the operator g which is an element of the group $\widehat{GL}(\infty)$ corresponding to the infinite dimensional Lie algebra $\widehat{gl}(\infty)$. The τ-function of the KP equation is sometimes defined as

$$\tau_{KP}(M, \mathbf{t}) = \langle M|e^{H(\mathbf{t})} g|M\rangle, \quad M \in Z, \tag{31}$$

where $\mathbf{t} = (t_1, t_2, \ldots)$ is the set of higher KP times. $H(\mathbf{t}) \in \widehat{gl}(\infty)$ is given by

$$H(\mathbf{t}) = \sum_{n=1}^{+\infty} t_n H_n, \quad H_n = \frac{1}{2\pi i} \oint : z^n \psi(z) \psi^*(z) : dz.$$

According to [5],[14] the integer M in (31) plays the role of discrete Toda lattice variable and defines the following charged vacuums

$$\langle M| = \langle 0|\Psi_M^*, \qquad |M\rangle = \Psi_M|0\rangle,$$

$$\begin{array}{llll} \Psi_M = \psi_{M-1} \cdots \psi_1 \psi_0 & M > 0, & \Psi_M = \psi_M^* \cdots \psi_{-2}^* \psi_{-1}^* & M < 0, \\ \Psi_M^* = \psi_0^* \psi_1^* \cdots \psi_{M-1}^* & M > 0, & \Psi_M^* = \psi_{-1} \psi_{-2} \cdots \psi_M & M < 0. \end{array}$$

The Baker-Akhiezer function and the conjugated one are

$$w(M, \mathbf{t}; k) = \frac{\langle M+1|e^{H(\mathbf{t})} \psi(k) g|M\rangle}{\langle M|e^{H(\mathbf{t})} g|M\rangle}, \tag{32}$$

$$w^*(M, \mathbf{t}; k) = \frac{\langle M-1|e^{H(\mathbf{t})} \psi^*(k) g|M\rangle}{\langle M|e^{H(\mathbf{t})} g|M\rangle}. \tag{33}$$

5.1 Trigonometric Flows

Let us consider the linear combinations of fermionic fields:

$$A_i = \oint c_i(k)\psi(k)dk, \quad A_i^* = \oint c_i^*(k)\psi^*(k)dk, \quad i = 1, 2, \ldots, N, \tag{34}$$

which satisfy the relations

$$A_i A_j^* + A_j^* A_i = h\delta_{ij}, \tag{35}$$

where $h \in C$ is a finite number, $c_i(k), c_i^*(k) \in C$.

Then we introduce a τ-function which depends on a time β in the following way:

$$\tau(M, \mathbf{t}, \beta) = \langle M|e^{H(\mathbf{t})} \prod_{i=1}^{N} e^{\beta A_i A_i^*} g|M\rangle. \tag{36}$$

This dependence describes a special one-parametric Backlund transformation of the τ-function $\tau(M, \mathbf{t}, \beta = 0)$. This is the fermionic version of N-fold "Zakharov-Shabat dressing depending on functional parameters" [6] of a given (nonvacuum) solution, see [9].

Proposition 5.1 *$\tau(M, \mathbf{t}, \beta)$ is a trigonometric function of β.*

Proof. For each pair $A = A_i,\ A^* = A_i^*$ we have

$$e^{\beta AA^*} = 1 + \beta AA^* + \frac{\beta^2}{2!} AA^*AA^* + ... = 1 + AA^* \frac{e^{\beta h} - 1}{h}. \tag{37}$$

Thus τ-function is a trigonometric function of β.

5.2 Trigonometric Calogero-Moser System (TCMS)

Proposition 5.2 *Let us impose the following symmetry constraint ($x = t_1$):*

$$\frac{\partial \tau}{\partial \beta} = \frac{\partial \tau}{\partial x}. \tag{38}$$

Then the motion of zeroes of $\tau(M, \mathbf{t}, \beta)$ in the x-plane is described by trigonometric Calogero-Moser system.

Proof. It follows from (38) that τ-function (36) is a trigonometric function of the variable x. In turn it is known [12], [13] that zeroes of τ-function which is the rational expression of e^{xh}, h is a constant, are described by RCMS.

Now let

$$a_i(\mathbf{t}) = \oint c_i(k)w(\mathbf{t};k)dk, \quad a_i^*(\mathbf{t}) = \oint c_i^*(k)w^*(\mathbf{t};k)dk, \quad i = 1, 2, \ldots, N. \tag{39}$$

Proposition 5.3 *Due to the formulae (39), (32), (33) the τ-function (36), (38) is also the τ-function of the following multicomponent Schrödinger equation*

$$(\partial_{t_2} - \partial_{t_1}^2 - \sum_i^N a_i a_i^*)a_j = 0, \quad (\partial_{t_2} + \partial_{t_1}^2 + \sum_i^N a_i a_i^*)a_j^* = 0. \tag{40}$$

See [9] for proof.

It follows from the results of [8],[9] that β can be identified with Θ of *Sections 3,4*. Compare also Proposition 4.1 and Proposition 5.2

The open problem is the problem of completeness of TCMS solutions which we get via the symmetry reduction among all TCMS solutions.

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